The explicit descriptions of the ramification loci for the problems of Goldberg (Developments in Computer Algebra Research)

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The explicit descriptions of the ramification loci for the problems of Goldberg

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Abstract

In this paper, we introduce the generalized Bell representation, and solve a problem of Goldberg that determine the number of equivalence classes of rational maps corresponding to each critical set, when the degree is small. Moreover, we consider the homogenization of the space of the critical set, and give simple expressions of singular loci.

1 Introduction

In [4], Goldberg suggested a problem that determine the number of equivalence classes of rational maps corresponding to each critical set. This problem is based on her theorem (Theorem 1.3 in [4]), and it is known that this theorem deeply concern with B. and M. Shapiro conjecture (see [1]).

As a joint work with M. Karima (Kabur Univ.) and M. Taniguchi (Nara Women's Univ.), we solve a problem of Goldberg when the degree is small (see [2] and [3]). In this paper, after summarizing the results in [3], we consider the homogenization of this problem and give defining equations of singular loci such as the exceptional loci or ramification loci explicitly.

A rational map of degree d is a map with the following form,

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are coprime polynomials with $\max\{\deg P, \deg Q\} = d$.

Definition 1

Two rational maps R_1 and R_2 are said to be *Möbius equivalent* if there is a Möbius transformation $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $R_2 = M \circ R_1$.

Let X_d be the set of all equivalence classes of rational maps of degree d, and $X_d^{(k)}$ be the set of classes of rational maps having critical point at ∞ with multiplicity k, where $k = 0$ means that ∞ is non-critical.

Remark 1

A rational map R of degree d has $2d - 2$ critical points counted including multiplicity. The set of critical points of R is invariant under taking a Möbius conjugate.

For each rational map R of degree d, the multiplicity of critical point at ∞ is at most $d-1$. Therefore, the space X_d is the disjoint union of $X_d^{(0)}, X_d^{(1)}, \cdots, X_d^{(d-1)}$.

Goldberg showed the following theorem.

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Theorem 2 (Goldberg [4])
A $(2d-2)$-tuple B is the critical set of at most $C(d)$ classes in X_d, where $C(d)$ means the d-th Catalan number $\frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}$. The maximal is attained by a Zariski open subset of the space $\hat{\mathbb{C}}^{2d-2}$ of all B.

The map $\Phi_d : X_d \to \widehat{\mathbb{C}}^{2d-2}$ is defined by sending a equivalence class to the set of critical points, and the restriction of Φ_d to $X_d^{(k)}$ is denoted by $\Phi_d^{(k)}$.

Then Goldberg's problem (see [4]) is written as follows:

Problem 1

- *Describe in detail the ramification sets of the maps Φ_d.*
- *Given a critical set $\boldsymbol{\alpha}$, determine the number of points in the preimage $\Phi_d^{-1}(\boldsymbol{\alpha})$.*

The critical set is called *admissible* if every point has multiplicity at most $d-1$. She also asked in [4] whether every admissible set in $\mathbb{C}^{2d-2}$ is attained by some rational map of degree d.

In the next section, we give the complete answer to these problems for the case of $d = 3$ and 4. We use "risa/asir", a symbolic and algebraic computation system, to obtain the defining equations of the loci considered.

2 Generalized Bell family

In this section, we summarize the results in ([3]).

First, we give the following extended version of Proposition 5 in [2]. Let $CB_d^{(k)}$ $(k = 0, 1, \cdots, d-1)$ be the *generalized Bell locus* consisting of all $H + \hat{P}/Q$, for

$$\begin{aligned} H(z) &= z^{k+1} + c_k z^k + \cdots + c_1 z, \\ \hat{P}(z) &= a_{d-k-2} z^{d-k-2} + \cdots + a_0, \\ Q(z) &= z^{d-k-1} + b_{d-k-2} z^{d-k-2} + \cdots + b_0, \end{aligned}$$

with $\mathrm{Resul}_z(\hat{P}, Q) \neq 0$.

Remark 2
If $k = d-1$, the generalized Bell locus is the family of polynomial maps $CB_d^{(d-1)} = \{z^d + c_{d-1}z^{d-1} + \cdots + c_1 z\}$. If $k = 0$, the generalized Bell locus coincides with the Bell locus; $CB_d^{(0)} = CB_d$ (see [2]).

Proposition 3
For every $R \in CB_d^{(k)}$, $[R]$ belongs to $X_d^{(k)}$ for every k, and for each element $[S]$ in $X_d^{(k)}$, there is a unique R in $CB_d^{(k)}$ with $[R] = [S]$.

Hence, each locus $X_d^{(k)}$ has a system of coordinates consisting of coefficients of representatives R in the generalized Bell locus $CB_d^{(k)}$.

Now, consider the map $\Phi_d^{(k)}$ of $CB_d^{(k)}$ to $\mathbb{C}^{2d-2-k}$ defined from the equation

$$\begin{aligned} \frac{1}{k+1}&\Big\{H'(z)Q^2(z) + \hat{P}'(z)Q(z) - \hat{P}(z)Q'(z)\Big\} \\ &= z^{2d-k-2} + \alpha_{2d-k-3} z^{2d-k-3} + \cdots + \alpha_0 = 0 \end{aligned}$$

by sending

$$(\mathbf{c}, \mathbf{a}, \mathbf{b}) = (c_k, \cdots, c_1,\ a_{d-k-2}, \cdots, a_0,\ b_{d-k-2}, \cdots, b_0)$$

to

$$\boldsymbol{\alpha} = (\alpha_{2d-k-3}, \cdots, \alpha_0).$$

Set
$$R_d^{(k)} = \{(\mathbf{c}, \mathbf{a}, \mathbf{b}) \in \mathbb{C}^{2d-2-k} \ : \ \mathrm{Resul}_z(\hat{P}, Q) = 0\},$$
which is the locus where $\Phi_d^{(k)}$ is not defined. (In other words, $CB_d^{(k)}$ can be identified with $\mathbb{C}^{2d-2-k} - R_d^{(k)}$.)

Here, we recall the following results in [2].

Proposition 4
The map $\Phi_2^{(0)} : CB_2^{(0)} \to \mathbb{C}^2 - E^{(0)}(2)$ is bijective, and the exceptional locus $E^{(0)}(2)$ is the algebraic curve defined by $\alpha_1^2 - 4\alpha_0 = 0$. And the map $\Phi_2^{(1)} : CB_2^{(1)} \to \mathbb{C}$ is bijective.

2.1 The case of degree 3 and 4

Now, we recall the following results in [2] and [3].

Proposition 5
The ramification locus of $\Phi_3^{(0)}$ is $a_1 = b_1^2 - 4b_0$, $\Phi_3^{(0)}(CB_3^{(0)}) = \mathbb{C}^4 - E^{(0)}(3)$, and $\Phi_3^{(0)}$ is 2-valent on the set of points in $\mathbb{C}^4 - E^{(0)}(3)$ satisfying that

$$\alpha_2^2 - 3\alpha_1\alpha_3 + 12\alpha_0 \neq 0, \quad E_0 \neq 0.$$

Here, the exceptional locus $E^{(0)}(3)$ is the algebraic variety defined by $E_0 = E_1 = 0$. Here

$$\begin{aligned} E_1 =& 27\alpha_1^2 - 9\alpha_2\alpha_3\alpha_1 + (27\alpha_3^2 - 72\alpha_2)\alpha_0 + 2\alpha_2^3, & (1)\\ E_0 =& -27\alpha_1^4 + (-4\alpha_3^3 + 18\alpha_2\alpha_3)\alpha_1^3 + ((-6\alpha_3^2 + 144\alpha_2)\alpha_0 + \alpha_2^2\alpha_3^2 - 4\alpha_2^3)\alpha_1^2 \\ &+ (-192\alpha_3\alpha_0^2 + (18\alpha_2\alpha_3^3 - 80\alpha_2^2\alpha_3)\alpha_0)\alpha_1 + 256\alpha_0^3 \\ &+ (-27\alpha_3^4 + 144\alpha_2\alpha_3^2 - 128\alpha_2^2)\alpha_0^2 + (-4\alpha_2^3\alpha_3^2 + 16\alpha_2^4)\alpha_0. & (2) \end{aligned}$$

In case $d = 3$, there remain the cases that ∞ is a critical point.

Proposition 6
The ramification locus of $\Phi_3^{(1)}$ is given by $c_1 - 2b_0 = 0$, $\Phi_3^{(1)}(CB_3^{(1)}) = \mathbb{C}^3 - E^{(1)}(3)$ and $\Phi_3^{(1)}$ is 2-valent on the the set of the points in $\mathbb{C}^3 - E^{(1)}(3)$ satisfying that

$$3\alpha_1 - \alpha_2^2 \neq 0, \quad 4\alpha_1^3 - \alpha_2^2\alpha_1^2 - 18\alpha_0\alpha_2\alpha_1 + 4\alpha_0\alpha_2^3 + 27\alpha_0^2 \neq 0.$$

Here, the exceptional locus $E^{(1)}(3)$ is the algebraic variety defined by

$$\{3\alpha_1 - \alpha_2^2 = 0, \quad 9\alpha_2\alpha_1 - 2\alpha_2^3 - 27\alpha_0 = 0\}.$$

Since the map $\Phi_3^{(2)} : CB_3^{(2)} \to \mathbb{C}^2$ is clearly bijective, we have obtained complete description for the case that $d = 3$.

Now, we summarize the results in the case of degree 4 ([3]).

Proposition 7
The ramification locus of $\Phi_4^{(0)}$ is given by

$$\begin{aligned} &(b_2a_2 - b_2^3 + 4b_1b_2 - 9b_0)a_1 - b_1a_2^2 + (-3a_0 + b_1b_2^2 + 6b_0b_2 - 5b_1^2)a_2 \\ &+ (b_2^2 - 3b_1)a_0 - 4b_0b_2^3 + b_1^2b_2^2 + 18b_0b_1b_2 - 4b_1^3 - 27b_0^2 = 0. \end{aligned}$$

The exceptional locus $E^{(0)}(4)$ is the algebraic variety defined by

$$
\begin{aligned}
\Big\{ & 6400\alpha_2^3 + (-9600\alpha_5\alpha_3 - 7680\alpha_4^2 + 9600\alpha_5^2\alpha_4 - 2000\alpha_5^4)\alpha_2^2 + ((12960\alpha_4 - 600\alpha_5^2)\alpha_3^2 \\
& \quad + (-9600\alpha_5\alpha_4^2 + 2400\alpha_5^3\alpha_4)\alpha_3 + 2304\alpha_4^4 - 640\alpha_5^2\alpha_4^3)\alpha_2 - 3645\alpha_3^4 + (3240\alpha_5\alpha_4 \\
& \quad - 800\alpha_5^3)\alpha_3^3 + (-864\alpha_4^3 + 240\alpha_5^2\alpha_4^2)\alpha_3^2 = 0, \\
& (2700\alpha_3 - 1800\alpha_5\alpha_4 + 500\alpha_5^3)\alpha_1 - 960\alpha_2^2 + (60\alpha_5\alpha_3 + 576\alpha_4^2 - 200\alpha_5^2\alpha_4)\alpha_2 \\
& \quad + (-216\alpha_4 + 75\alpha_5^2)\alpha_3^2 = 0, \\
& (-10800\alpha_2 + 6480\alpha_4^2 - 3600\alpha_5^2\alpha_4 + 500\alpha_5^4)\alpha_1 + 5520\alpha_5\alpha_2^2 + ((-6912\alpha_4 + 60\alpha_5^2)\alpha_3 \\
& \quad + 1296\alpha_5\alpha_4^2 - 200\alpha_5^3\alpha_4)\alpha_2 + 2187\alpha_3^3 + (-486\alpha_5\alpha_4 + 75\alpha_5^3)\alpha_3^2 = 0, \\
& -135000\alpha_1^2 + (54000\alpha_5\alpha_4^2 - 40000\alpha_5^3\alpha_4 + 7500\alpha_5^5)\alpha_1 + (-26880\alpha_4 + 34200\alpha_5^2)\alpha_2^2 \\
& \quad + (9720\alpha_3^2 + (-43680\alpha_5\alpha_4 + 900\alpha_5^3)\alpha_3 - 2304\alpha_4^3 + 13840\alpha_5^2\alpha_4^2 - 3000\alpha_5^4\alpha_4)\alpha_2 \\
& \quad + 13365\alpha_5\alpha_3^3 + (864\alpha_4^2 - 5190\alpha_5^2\alpha_4 + 1125\alpha_5^4)\alpha_3^2 = 0, \\
& -20\alpha_5\alpha_1 + 8\alpha_4\alpha_2 - 3\alpha_3^2 + 120\alpha_0 = 0 \Big\}.
\end{aligned}
\tag{3}
$$

Moreover, for a given $\boldsymbol{\alpha}$ in $\mathbb{C}^6 - E^{(0)}(4)$, b_1 is a solution of algebraic equation of degree 5, and other coefficients are determined from b_1 and $\boldsymbol{\alpha}$. In particular, $\Phi_4^{(0)}$ is 5-valent on the set of points in $\mathbb{C}^6 - E^{(0)}(4)$ satisfying $E_0^{(0)} \neq 0$ and $D^{(0)} \neq 0$.

Here, $E_0^{(0)} = 0$ gives the locus where the numerator and the denominator of R has a non-constant common factor, and $D^{(0)}$ is the discriminant of the equation whose solution gives the coefficient b_1.

$$
\begin{aligned}
E_0^{(0)} = {} & 3125\alpha_0^4\alpha_5^6 + (-2500\alpha_0^3\alpha_1\alpha_4 + (-3750\alpha_0^3\alpha_2 + 2000\alpha_0^2\alpha_1^2)\alpha_3 + 2250\alpha_0^2\alpha_1\alpha_2^2 - 1600\alpha_0\alpha_1^3\alpha_2 \\
& +256\alpha_1^5)\alpha_5^5 + ((2000\alpha_0^3\alpha_2 - 50\alpha_0^2\alpha_1^2)\alpha_4^2 + (2250\alpha_0^3\alpha_3^2 + (-2050\alpha_0^2\alpha_1\alpha_2 + 160\alpha_0\alpha_1^3)\alpha_3 - 900\alpha_0^2\alpha_2^3 \\
& +1020\alpha_0\alpha_1^2\alpha_2^2 - 192\alpha_1^4\alpha_2 - 22500\alpha_0^4)\alpha_4 - 900\alpha_0^2\alpha_1\alpha_3^3 + (825\alpha_0^2\alpha_2^2 + 560\alpha_0\alpha_1^2\alpha_2 - 128\alpha_1^4)\alpha_3^2 + (-630\alpha_0\alpha_1\alpha_2^3 \\
& +144\alpha_1^3\alpha_2^2 + 2250\alpha_0^3\alpha_1)\alpha_3 + 108\alpha_0\alpha_2^5 - 27\alpha_1^2\alpha_2^4 + 1500\alpha_0^3\alpha_2^2 - 1700\alpha_0^2\alpha_1^2\alpha_2 + 320\alpha_0\alpha_1^4)\alpha_5^4 + ((-1600\alpha_0^3\alpha_3 \\
& +160\alpha_0^2\alpha_1\alpha_2 - 36\alpha_0\alpha_1^3)\alpha_4^3 + (1020\alpha_0^2\alpha_1\alpha_3^2 + (560\alpha_0^2\alpha_2^2 - 746\alpha_0\alpha_1^2\alpha_2 + 144\alpha_1^4)\alpha_3 + 24\alpha_0\alpha_1\alpha_2^3 - 6\alpha_1^3\alpha_2^2 \\
& +15600\alpha_0^3\alpha_1)\alpha_4^2 + ((-630\alpha_0^2\alpha_2 + 24\alpha_0\alpha_1^2)\alpha_3^3 + (356\alpha_0\alpha_1\alpha_2^2 - 80\alpha_1^3\alpha_2)\alpha_3^2 - (72\alpha_0\alpha_2^4 - 18\alpha_1^2\alpha_2^3 - 19800\alpha_0^3\alpha_2 \\
& +12330\alpha_0^2\alpha_1^2)\alpha_3 - 13040\alpha_0^2\alpha_1\alpha_2^2 + 9768\alpha_0\alpha_1^3\alpha_2 - 1600\alpha_1^5)\alpha_4 + 108\alpha_0^2\alpha_3^5 + (-72\alpha_0\alpha_1\alpha_2 + 16\alpha_1^3)\alpha_3^4 \\
& +(16\alpha_0\alpha_2^3 - 4\alpha_1^2\alpha_2^2 - 1350\alpha_0^3)\alpha_3^3 + (1980\alpha_0^2\alpha_1\alpha_2 - 208\alpha_0\alpha_1^3)\alpha_3^2 + (-120\alpha_0^2\alpha_2^3 - 682\alpha_0\alpha_1^2\alpha_2^2 + 160\alpha_1^4\alpha_2 \\
& +27000\alpha_0^4)\alpha_3 + 144\alpha_0\alpha_1\alpha_2^4 - 36\alpha_1^3\alpha_2^3 - 1800\alpha_0^3\alpha_1\alpha_2 + 410\alpha_0^2\alpha_1^3)\alpha_5^3 + (256\alpha_0^3\alpha_4^5 + (-192\alpha_0^2\alpha_1\alpha_3 - 128\alpha_0^2\alpha_2^2 \\
& +144\alpha_0\alpha_1^2\alpha_2 - 27\alpha_1^4)\alpha_4^4 + ((144\alpha_0^2\alpha_2 - 6\alpha_0\alpha_1^2)\alpha_3^2 - (80\alpha_0\alpha_1\alpha_2^2 - 18\alpha_1^3\alpha_2)\alpha_3 + 16\alpha_0\alpha_2^4 - 4\alpha_1^2\alpha_2^3 - 10560\alpha_0^3\alpha_2 \\
& +248\alpha_0^2\alpha_1^2)\alpha_4^3 - (27\alpha_0^2\alpha_3^4 - (18\alpha_0\alpha_1\alpha_2 - 4\alpha_1^3)\alpha_3^3 + (4\alpha_0\alpha_2^3 - \alpha_1^2\alpha_2^2 + 9720\alpha_0^3)\alpha_3^2 - (10152\alpha_0^2\alpha_1\alpha_2 \\
& -682\alpha_0\alpha_1^3)\alpha_3 - 4816\alpha_0^2\alpha_2^3 + 5428\alpha_0\alpha_1^2\alpha_2^2 - 1020\alpha_1^4\alpha_2 - 43200\alpha_0^4)\alpha_4^2 + (3942\alpha_0^2\alpha_1\alpha_3^3 + (-4536\alpha_0^2\alpha_2^2 \\
& -2412\alpha_0\alpha_1^2\alpha_2 + 560\alpha_1^4)\alpha_3^2 + (3272\alpha_0\alpha_1\alpha_2^3 - 746\alpha_1^3\alpha_2^2 - 31320\alpha_0^3\alpha_1)\alpha_3 - 576\alpha_0\alpha_2^5 + 144\alpha_1^2\alpha_2^4 - 6480\alpha_0^3\alpha_2^2 \\
& +8748\alpha_0^2\alpha_1^2\alpha_2 - 1700\alpha_0\alpha_1^4)\alpha_4 + 162\alpha_0^2\alpha_2\alpha_3^4 + (-108\alpha_0\alpha_1\alpha_2^2 + 24\alpha_1^3\alpha_2)\alpha_3^3 + (24\alpha_0\alpha_2^4 - 6\alpha_1^2\alpha_2^3 - 27540\alpha_0^3\alpha_2 \\
& +15417\alpha_0^2\alpha_1^2)\alpha_3^2 + (16632\alpha_0^2\alpha_1\alpha_2^2 - 12330\alpha_0\alpha_1^3\alpha_2 + 2000\alpha_1^5)\alpha_3 - 192\alpha_0^2\alpha_2^4 + 248\alpha_0\alpha_1^2\alpha_2^3 - 50\alpha_1^4\alpha_2^2 \\
& -32400\alpha_0^4\alpha_2 + 540\alpha_0^3\alpha_1^2)\alpha_5^2 + ((6912\alpha_0^3\alpha_3 - 640\alpha_0^2\alpha_1\alpha_2 + 144\alpha_0\alpha_1^3)\alpha_4^4 + (-4464\alpha_0^2\alpha_1\alpha_3^2 + (-2496\alpha_0^2\alpha_2^2 \\
& +3272\alpha_0\alpha_1^2\alpha_2 - 630\alpha_1^4)\alpha_3 - 96\alpha_0\alpha_1\alpha_2^3 + 24\alpha_1^3\alpha_2^2 - 21888\alpha_0^3\alpha_1)\alpha_4^3 + ((2808\alpha_0^2\alpha_2 - 108\alpha_0\alpha_1^2)\alpha_3^3 \\
& +(-1584\alpha_0\alpha_1\alpha_2^2 + 356\alpha_1^3\alpha_2)\alpha_3^2 + (320\alpha_0\alpha_2^4 - 80\alpha_1^2\alpha_2^3 - 3456\alpha_0^3\alpha_2 + 16632\alpha_0^2\alpha_1^2)\alpha_3 + 15264\alpha_0^2\alpha_1\alpha_2^2 \\
& -13040\alpha_0\alpha_1^3\alpha_2 + 2250\alpha_1^5)\alpha_4^2 + (-486\alpha_0^2\alpha_3^5 + (324\alpha_0\alpha_1\alpha_2 - 72\alpha_1^3)\alpha_3^4 + (-72\alpha_0\alpha_2^3 + 18\alpha_1^2\alpha_2^2 + 21384\alpha_0^3)\alpha_3^3 \\
& +(-22896\alpha_0^2\alpha_1\alpha_2 + 1980\alpha_0\alpha_1^3)\alpha_3^2 + (-5760\alpha_0^2\alpha_2^3 + 10152\alpha_0\alpha_1^2\alpha_2^2 - 2050\alpha_1^4\alpha_2 - 77760\alpha_0^4)\alpha_3 - 640\alpha_0\alpha_1\alpha_2^4 \\
& +160\alpha_1^3\alpha_2^3 + 31968\alpha_0^3\alpha_1\alpha_2 - 1800\alpha_0^2\alpha_1^3)\alpha_4 - 6318\alpha_0^2\alpha_1\alpha_3^4 + (5832\alpha_0^2\alpha_2^2 + 3942\alpha_0\alpha_1^2\alpha_2 - 900\alpha_1^4)\alpha_3^3 \\
& +(-4464\alpha_0\alpha_1\alpha_2^3 + 1020\alpha_1^3\alpha_2^2 + 15552\alpha_0^3\alpha_1)\alpha_3^2 + (768\alpha_0\alpha_2^5 - 192\alpha_1^2\alpha_2^4 + 46656\alpha_0^3\alpha_2^2 - 31320\alpha_0^2\alpha_1^2\alpha_2 \\
& +2250\alpha_0\alpha_1^4)\alpha_3 - 21888\alpha_0^2\alpha_1\alpha_2^3 + 15600\alpha_0\alpha_1^3\alpha_2^2 - 2500\alpha_1^5\alpha_2 + 38880\alpha_0^4\alpha_1)\alpha_5 - 1024\alpha_0^3\alpha_4^6 + (768\alpha_0^2\alpha_1\alpha_3 \\
& +512\alpha_0^2\alpha_2^2 - 576\alpha_0\alpha_1^2\alpha_2 + 108\alpha_1^4)\alpha_4^5 + ((-576\alpha_0^2\alpha_2 + 24\alpha_0\alpha_1^2)\alpha_3^2 + (320\alpha_0\alpha_1\alpha_2^2 - 72\alpha_1^3\alpha_2)\alpha_3 - 64\alpha_0\alpha_2^4 \\
& +16\alpha_1^2\alpha_2^3 + 9216\alpha_0^3\alpha_2 - 192\alpha_0^2\alpha_1^2)\alpha_4^4 + (108\alpha_0^2\alpha_3^4 + (-72\alpha_0\alpha_1\alpha_2 + 16\alpha_1^3)\alpha_3^3 + (16\alpha_0\alpha_2^3 - 4\alpha_1^2\alpha_2^2 - 8640\alpha_0^3)\alpha_3^2
\end{aligned}
$$

$$+(-5760\alpha_0^2\alpha_1\alpha_2 - 120\alpha_0\alpha_1^3)\alpha_3 - 4352\alpha_0^2\alpha_2^3 + 4816\alpha_0\alpha_1^2\alpha_2^2 - 900\alpha_1^4\alpha_2 - 13824\alpha_0^4)\alpha_4^3 + (5832\alpha_0^2\alpha_1\alpha_3^3$$
$$+(8208\alpha_0^2\alpha_2^2 - 4536\alpha_0\alpha_1^2\alpha_2 + 825\alpha_1^4)\alpha_3^2 + (-2496\alpha_0\alpha_1\alpha_2^3 + 560\alpha_1^3\alpha_2^2 + 46656\alpha_0^3\alpha_1)\alpha_3 + 512\alpha_0\alpha_2^5 - 128\alpha_1^2\alpha_2^4$$
$$-17280\alpha_0^3\alpha_2^2 - 6480\alpha_0^2\alpha_1^2\alpha_2 + 1500\alpha_0\alpha_1^4)\alpha_4^2 + ((-4860\alpha_0^2\alpha_2 + 162\alpha_0\alpha_1^2)\alpha_3^4 + (2808\alpha_0\alpha_1\alpha_2^2 - 630\alpha_1^3\alpha_2)\alpha_3^3$$
$$+(-576\alpha_0\alpha_2^4 + 144\alpha_1^2\alpha_2^3 + 3888\alpha_0^3\alpha_2 - 27540\alpha_0^2\alpha_1^2)\alpha_3^2 + (-3456\alpha_0^2\alpha_1\alpha_2^2 + 19800\alpha_0\alpha_1^3\alpha_2 - 3750\alpha_1^5)\alpha_3$$
$$+9216\alpha_0^2\alpha_2^4 - 10560\alpha_0\alpha_1^2\alpha_2^3 + 2000\alpha_1^4\alpha_2^2 + 62208\alpha_0^4\alpha_2 - 32400\alpha_0^3\alpha_1^2)\alpha_4 + 729\alpha_0^2\alpha_3^6 + (-486\alpha_0\alpha_1\alpha_2$$
$$+108\alpha_1^3)\alpha_3^5 + (108\alpha_0\alpha_2^3 - 27\alpha_1^2\alpha_2^2 - 8748\alpha_0^3)\alpha_3^4 + (21384\alpha_0^2\alpha_1\alpha_2 - 1350\alpha_0\alpha_1^3)\alpha_3^3 - (8640\alpha_0^2\alpha_2^3 + 9720\alpha_0\alpha_1^2\alpha_2^2$$
$$-2250\alpha_1^4\alpha_2 - 34992\alpha_0^4)\alpha_3^2 + (6912\alpha_0\alpha_1\alpha_2^4 - 1600\alpha_1^3\alpha_2^3 - 77760\alpha_0^3\alpha_1\alpha_2 + 27000\alpha_0^2\alpha_1^3)\alpha_3 - 1024\alpha_0\alpha_2^6$$
$$+256\alpha_1^2\alpha_2^5 - 13824\alpha_0^3\alpha_2^3 + 43200\alpha_0^2\alpha_1^2\alpha_2^2 - 22500\alpha_0\alpha_1^4\alpha_2 + 3125\alpha_1^6 - 46656\alpha_0^5.$$

$$D^{(0)} = (6\alpha_5^3\alpha_4 - \alpha_5^5 - 27\alpha_3\alpha_5^2 + 108\alpha_2\alpha_5 - 324\alpha_1)^2$$
$$\times 12(108\alpha_0\alpha_1^2\alpha_4 - 648\alpha_0^2\alpha_1\alpha_5 + (-108\alpha_0\alpha_1\alpha_2 - 27\alpha_1^3)\alpha_3 + 32\alpha_0\alpha_2^3 + 9\alpha_1^2\alpha_2^2)\alpha_4^6 + 4(2916\alpha_0^3\alpha_5^2 + ((972\alpha_0^2\alpha_2$$
$$-1863\alpha_0\alpha_1^2)\alpha_3 - 234\alpha_0\alpha_1\alpha_2^2 + 27\alpha_1^3\alpha_2)\alpha_5 + 81\alpha_0\alpha_1\alpha_3^3 + (-27\alpha_0\alpha_2^2 + 81\alpha_1^2\alpha_2)\alpha_3^2 - (51\alpha_1\alpha_2^3 - 2916\alpha_0^2\alpha_1)\alpha_3$$
$$+8\alpha_2^5 - 2592\alpha_0^2\alpha_2^2 + 5022\alpha_0\alpha_1^2\alpha_2 - 162\alpha_1^4)\alpha_4^5 + (49572\alpha_0^2\alpha_1\alpha_3 + 108\alpha_0^2\alpha_2^2 + 4284\alpha_0\alpha_1^2\alpha_2 + 27\alpha_1^4)\alpha_5^2$$
$$+(-972\alpha_0^2\alpha_3^3 + (8316\alpha_0\alpha_1\alpha_2 + 2106\alpha_1^3)\alpha_3^2 + (-2412\alpha_0\alpha_2^3 - 738\alpha_1^2\alpha_2^2 - 34992\alpha_0^3)\alpha_3 + 8\alpha_1\alpha_2^4 - 92016\alpha_0^2\alpha_1\alpha_2$$
$$-42444\alpha_0\alpha_1^3)\alpha_5 + -81\alpha_1^2\alpha_3^4 + 54\alpha_1\alpha_2^2\alpha_3^3 + (-9\alpha_2^4 + 1944\alpha_0^2\alpha_2 - 4860\alpha_0\alpha_1^2)\alpha_3^2(-13500\alpha_0\alpha_1\alpha_2^2$$
$$-3888\alpha_1^3\alpha_2)\alpha_3 + 4320\alpha_0\alpha_2^4 + 1320\alpha_1^2\alpha_2^3 + 93312\alpha_0^3\alpha_2 - 329508\alpha_0^2\alpha_1^2)\alpha_4^4 - 4((18225\alpha_0^3\alpha_3 + 6615\alpha_0^2\alpha_1\alpha_2$$
$$-849\alpha_0\alpha_1^3)\alpha_5^3 + ((6156\alpha_0^2\alpha_2 - 2322\alpha_0\alpha_1^2)\alpha_3^2 + (-378\alpha_0\alpha_1\alpha_2^2 + 450\alpha_1^3\alpha_2)\alpha_3 - 330\alpha_0\alpha_2^4 - 92\alpha_1^2\alpha_2^3$$
$$-31590\alpha_0^3\alpha_2 - 117693\alpha_0^2\alpha_1^2)\alpha_5^2 + (486\alpha_0\alpha_1\alpha_3^4 + (-162\alpha_0\alpha_2^2 + 513\alpha_1^2\alpha_2)\alpha_3^3 + (-324\alpha_1\alpha_2^3 + 15795\alpha_0^2\alpha_1)\alpha_3^2$$
$$+(51\alpha_2^5 - 27621\alpha_0^2\alpha_2^2 + 7182\alpha_0\alpha_1^2\alpha_2 - 3645\alpha_1^4)\alpha_3 + 6030\alpha_0\alpha_1\alpha_2^3 + 531\alpha_1^3\alpha_2^2 - 435942\alpha_0^3\alpha_1)\alpha_5$$
$$+(-972\alpha_0\alpha_1\alpha_2 - 81\alpha_1^3)\alpha_3^3 + (324\alpha_0\alpha_2^3 - 918\alpha_1^2\alpha_2^2 + 2187\alpha_0^3)\alpha_3^2 + (603\alpha_1\alpha_2^4 - 99387\alpha_0^2\alpha_1\alpha_2 - 57105\alpha_0\alpha_1^3)\alpha_3$$
$$-96\alpha_2^6 + 50544\alpha_0^2\alpha_2^3 - 6696\alpha_0\alpha_1^2\alpha_2^2 + 2025\alpha_1^4\alpha_2 + 69984\alpha_0^4)\alpha_4^3 + 2((20250\alpha_0^3\alpha_2 - 24975\alpha_0^2\alpha_1^2)\alpha_5^4$$
$$+(-43335\alpha_0^2\alpha_1\alpha_3^2 + (6345\alpha_0^2\alpha_2^2 - 6642\alpha_0\alpha_1^2\alpha_2 - 492\alpha_1^4)\alpha_3 - 1062\alpha_0\alpha_1\alpha_2^3 + 317\alpha_1^3\alpha_2^2 - 814050\alpha_0^3\alpha_1)\alpha_5^3$$
$$+(2916\alpha_0^2\alpha_3^4 + (-6804\alpha_0\alpha_1\alpha_2 - 1917\alpha_1^3)\alpha_3^3 + (1836\alpha_0\alpha_2^3 + 1296\alpha_1^2\alpha_2^2 + 102060\alpha_0^3)\alpha_3^2 + (-369\alpha_1\alpha_2^4$$
$$+6804\alpha_0^2\alpha_1\alpha_2 - 19224\alpha_0\alpha_1^3)\alpha_3 + 54\alpha_2^6 + 13392\alpha_0^2\alpha_2^3 + 57672\alpha_0\alpha_1^2\alpha_2^2 - 2610\alpha_1^4\alpha_2 - 1202850\alpha_0^4)\alpha_5^2$$
$$+(243\alpha_1^2\alpha_3^5 - 162\alpha_1\alpha_2^2\alpha_3^4 + (27\alpha_2^4 - 11664\alpha_0^2\alpha_2 + 8262\alpha_0\alpha_1^2)\alpha_3^3 + (22680\alpha_0\alpha_1\alpha_2^2 + 1512\alpha_1^3\alpha_2)\alpha_3^2$$
$$+(-6750\alpha_0\alpha_2^4 + 756\alpha_1^2\alpha_2^3 - 858762\alpha_0^3\alpha_2 - 536301\alpha_0^2\alpha_1^2)\alpha_3 - 468\alpha_1\alpha_2^5 + 280260\alpha_0^2\alpha_1\alpha_2^2 - 246780\alpha_0\alpha_1^3\alpha_2$$
$$+10125\alpha_1^5)\alpha_5 - 486\alpha_1^2\alpha_2\alpha_3^4 + (324\alpha_1\alpha_2^3 - 32805\alpha_0^2\alpha_1)\alpha_3^3 + (-54\alpha_2^5 + 22599\alpha_0^2\alpha_2^2 - 134622\alpha_0\alpha_1^2\alpha_2$$
$$-18225\alpha_1^4)\alpha_3^2 + (55242\alpha_0\alpha_1\alpha_2^3 + 6345\alpha_1^3\alpha_2^2 - 1167858\alpha_0^3\alpha_1)\alpha_3 - 5184\alpha_0\alpha_2^5 + 54\alpha_1^2\alpha_2^4 + 1562976\alpha_0^3\alpha_2^2$$
$$-733860\alpha_0^2\alpha_1^2\alpha_2 + 344250\alpha_0\alpha_1^4)\alpha_4^2 + 4(50625\alpha_0^3\alpha_1\alpha_5^5 + (30375\alpha_0^3\alpha_3^2 + (29700\alpha_0^2\alpha_1\alpha_2 + 3060\alpha_0\alpha_1^3)\alpha_3$$
$$-2025\alpha_0^2\alpha_2^3 - 1305\alpha_0\alpha_1^2\alpha_2^2 + 88\alpha_1^4\alpha_2 + 455625\alpha_0^4)\alpha_5^4 + ((9720\alpha_0^2\alpha_2 + 1539\alpha_0\alpha_1^2)\alpha_3^3 + (756\alpha_0\alpha_1\alpha_2^2$$
$$+1386\alpha_1^3\alpha_2)\alpha_3^2 + (-972\alpha_0\alpha_2^4 - 450\alpha_1^2\alpha_2^3 + 24300\alpha_0^3\alpha_2 - 61155\alpha_0^2\alpha_1^2)\alpha_3 + 27\alpha_1\alpha_2^5 - 123390\alpha_0^2\alpha_1\alpha_2^2$$
$$+4698\alpha_0\alpha_1^3\alpha_2 - 600\alpha_1^5)\alpha_5^3 + (729\alpha_0\alpha_1\alpha_3^5 + (-243\alpha_0\alpha_2^2 + 810\alpha_1^2\alpha_2)\alpha_3^4 + (-513\alpha_1\alpha_2^3 + 38637\alpha_0^2\alpha_1)\alpha_3^3$$
$$+(81\alpha_2^5 - 67311\alpha_0^2\alpha_2^2 + 5670\alpha_0\alpha_1^2\alpha_2 - 4590\alpha_1^4)\alpha_3^2 + (-7182\alpha_0\alpha_1\alpha_2^3 - 3321\alpha_1^3\alpha_2^2 + 798255\alpha_0^3\alpha_1)\alpha_3$$
$$+5022\alpha_0\alpha_2^5 + 1071\alpha_1^2\alpha_2^4 - 366930\alpha_0^3\alpha_2^2 + 748278\alpha_0^2\alpha_1^2\alpha_2 - 12825\alpha_0\alpha_1^4)\alpha_5^2 + ((-2916\alpha_0\alpha_1\alpha_2 + 243\alpha_1^3)\alpha_3^4$$
$$+(972\alpha_0\alpha_2^3 - 3402\alpha_1^2\alpha_2^2 + 59049\alpha_0^3)\alpha_3^3 + (2079\alpha_1\alpha_2^4 - 1458\alpha_0^2\alpha_1\alpha_2 + 20655\alpha_0\alpha_1^3)\alpha_3^2 + (-324\alpha_2^6$$
$$+99387\alpha_0^2\alpha_2^3 + 3402\alpha_0\alpha_1^2\alpha_2^2 + 29700\alpha_1^4\alpha_2 + 2066715\alpha_0^4)\alpha_3 - 23004\alpha_0\alpha_1\alpha_2^4 - 6615\alpha_1^3\alpha_2^3 - 538002\alpha_0^3\alpha_1\alpha_2$$
$$-1148175\alpha_0^2\alpha_1^3)\alpha_5 + 13122\alpha_0\alpha_1^2\alpha_3^4 + (-5832\alpha_0\alpha_1\alpha_2^2 + 9720\alpha_1^3\alpha_2)\alpha_3^3 + (486\alpha_0\alpha_2^4 - 6156\alpha_1^2\alpha_2^3 - 118098\alpha_0^3\alpha_2$$
$$+557685\alpha_0^2\alpha_1^2)\alpha_3^2 + (972\alpha_1\alpha_2^5 - 429381\alpha_0^2\alpha_1\alpha_2^2 + 24300\alpha_0\alpha_1^3\alpha_2 - 50625\alpha_1^5)\alpha_3 + 23328\alpha_0^2\alpha_2^4 + 31590\alpha_0\alpha_1^2\alpha_2^3$$
$$+10125\alpha_1^4\alpha_2^2 - 5196312\alpha_0^4\alpha_2 + 3936600\alpha_0^3\alpha_1^2)\alpha_4 - 253125\alpha_0^4\alpha_5^6 - 2((101250\alpha_0^3\alpha_2 + 13500\alpha_0^2\alpha_1^2)\alpha_3$$
$$-10125\alpha_0^2\alpha_1\alpha_2^2 + 1200\alpha_0\alpha_1^3\alpha_2 - 32\alpha_1^5)\alpha_5^5 + 3(4050\alpha_0^2\alpha_1\alpha_3^3 + (-12150\alpha_0^2\alpha_2^2 - 6120\alpha_0\alpha_1^2\alpha_2 - 144\alpha_1^4)\alpha_3^2$$
$$+(4860\alpha_0\alpha_1\alpha_2^3 - 328\alpha_1^3\alpha_2^2 + 243000\alpha_0^3\alpha_1)\alpha_3 - 216\alpha_0\alpha_2^5 + 9\alpha_1^2\alpha_2^4 + 229500\alpha_0^3\alpha_2^2 - 17100\alpha_0^2\alpha_1^2\alpha_2$$
$$+3360\alpha_0\alpha_1^4)\alpha_5^4 - 6(1458\alpha_0^2\alpha_3^5 + (-162\alpha_0\alpha_1\alpha_2 - 162\alpha_1^3)\alpha_3^4 + (-54\alpha_0\alpha_2^3 + 639\alpha_1^2\alpha_2^2 + 72900\alpha_0^3)\alpha_3^3$$
$$+(-351\alpha_1\alpha_2^4 - 13770\alpha_0^2\alpha_1\alpha_2 - 10908\alpha_0\alpha_1^3)\alpha_3^2 + (54\alpha_2^6 - 38070\alpha_0^2\alpha_2^3 + 6408\alpha_0\alpha_1^2\alpha_2^2 - 2040\alpha_1^4\alpha_2$$
$$+820125\alpha_0^4)\alpha_3 + 7074\alpha_0\alpha_1\alpha_2^4 - 566\alpha_1^3\alpha_2^3 + 765450\alpha_0^3\alpha_1\alpha_2 - 1215\alpha_0^2\alpha_1^3)\alpha_5^3 - 27(27\alpha_1^2\alpha_3^6 - 18\alpha_1\alpha_2^2\alpha_3^5$$
$$+(3\alpha_2^4 - 1944\alpha_0^2\alpha_2 + 1512\alpha_0\alpha_1^2)\alpha_3^4 + (-612\alpha_0\alpha_1\alpha_2^2 - 228\alpha_1^3\alpha_2)\alpha_3^3 + (180\alpha_0\alpha_2^4 - 344\alpha_1^2\alpha_2^3 - 82620\alpha_0^3\alpha_2$$
$$+15876\alpha_0^2\alpha_1^2)\alpha_3^2 + (276\alpha_1\alpha_2^5 + 39726\alpha_0^2\alpha_1\alpha_2^2 + 9060\alpha_0\alpha_1^3\alpha_2 + 1000\alpha_1^5)\alpha_3 - 48\alpha_2^7 + 12204\alpha_0^2\alpha_2^4$$
$$-17436\alpha_0\alpha_1^2\alpha_2^3 + 1850\alpha_1^4\alpha_2^2 - 583200\alpha_0^4\alpha_2 - 281880\alpha_0^3\alpha_1^2)\alpha_5^2 + 162(18\alpha_1^2\alpha_2\alpha_3^5 + (-12\alpha_1\alpha_2^3 - 729\alpha_0^2\alpha_1)\alpha_3^4$$
$$+(2\alpha_2^5 - 405\alpha_0^2\alpha_2^2 + 954\alpha_0\alpha_1^2\alpha_2 + 75\alpha_1^4)\alpha_3^3 + (-390\alpha_0\alpha_1\alpha_2^3 - 535\alpha_1^3\alpha_2^2 - 32076\alpha_0^3\alpha_1)\alpha_3^2 + (72\alpha_0\alpha_2^5 + 306\alpha_1^2\alpha_2^4$$

$$-14418\alpha_0^3\alpha_2^2 + 19710\alpha_0^2\alpha_1^2\alpha_2 + 4500\alpha_0\alpha_1^4)\alpha_3 - 48\alpha_1\alpha_2^6 + 10764\alpha_0^2\alpha_1\alpha_2^3 - 10050\alpha_0\alpha_1^3\alpha_2^2 + 1250\alpha_1^5\alpha_2$$
$$-262440\alpha_0^4\alpha_1)\alpha_5 - 81(108\alpha_1^3\alpha_3^5 - 72\alpha_1^2\alpha_2^2\alpha_3^4 + (12\alpha_1\alpha_2^4 - 2916\alpha_0^2\alpha_1\alpha_2 + 5400\alpha_0\alpha_1^3)\alpha_3^3 + (108\alpha_0^2\alpha_2^3$$
$$-2520\alpha_0\alpha_1^2\alpha_2^2 - 1500\alpha_1^4\alpha_2 - 19683\alpha_0^4)\alpha_3^2 + (432\alpha_0\alpha_1\alpha_2^4 + 900\alpha_1^3\alpha_2^3 - 102060\alpha_0^3\alpha_1\alpha_2 + 60750\alpha_0^2\alpha_1^3)\alpha_3$$
$$-144\alpha_1^2\alpha_2^5 + 3456\alpha_0^3\alpha_2^3 + 29700\alpha_0^2\alpha_1^2\alpha_2^2 - 22500\alpha_0\alpha_1^4\alpha_2 + 3125\alpha_1^6 - 629856\alpha_0^5).$$

The details of the number of preimages are shown in the following Table 1, where $E_k^{(0)}$ $(k = 0, \cdots, 4)$ mean the coefficients of equation

$$34828517376r^5 + 5038848E_4^{(0)}r^4 + 186624E_3^{(0)}r^3 - 864E_2^{(0)}r^2 + 16E_1^{(0)}E_0^{(0)}r - (E_0^{(0)})^2 = 0$$

obtained by eliminating $b_2, b_1, b_0, a_2, a_1, a_0$ from the resultant $r = \mathrm{Resul}_z(\widehat{P}, Q)$, and each $I_k^{(0)}$ means an algebraic variety that the defining equation is omitted here.

$\#(\Phi_4^{(0)})^{-1}(\alpha)$	α	Remark
0	$E^{(1)}(4)$	
1	$E_0^{(0)} = E_2^{(0)} = E_3^{(0)} = 0,\ E_4^{(0)} \neq 0$	inv=1
2	$E_0^{(0)} = E_2^{(0)} = 0,\ E_3^{(0)} \neq 0$	inv=2
3	$E_0^{(0)} = 0,\ E_2^{(0)} \neq 0$	inv=3
4	$\emptyset$	inv=4
1	$\emptyset$	5-ple
2	$I_4^{(0)}$	4-ple
2	$I_{3,2}^{(0)}$	double+triple
3	$I_3^{(0)}$	triple
3	$I_{2,2}^{(0)}$	double+double
4	$I_2^{(0)}$	double
5	otherwise	

Table 1: The number of inverse images.

If ∞ is a simple critical point, we have the following.

Proposition 8
The ramification locus of the map $\Phi_4^{(1)}$ is given by

$$2c_1b_1^3 - (c_1^2 + 4b_0)b_1^2 - (8b_0c_1 + 2a_1)b_1 + 4b_0c_1^2 + a_1c_1 + 2a_0 + 16b_0^2 = 0,$$

$\Phi_4^{(1)}(CB_4^{(1)}) = \mathbb{C}^5 - E^{(1)}(4)$, and $\Phi_4^{(1)}$ is 5-valent on the set of points in $\mathbb{C}^5 - E^{(1)}(4)$ satisfying $D^{(1)} \neq 0$ and $E_0^{(1)} \neq 0$, where $D^{(1)}$ and $E_0^{(1)}$ are given below. Moreover, the exceptional locus $E^{(1)}(4)$ is the algebraic variety defined by

$$\left\{ \begin{array}{l} 25\alpha_2^2 + (-30\alpha_4\alpha_3 + 8\alpha_4^3)\alpha_2 + 10\alpha_3^3 - 3\alpha_4^2\alpha_3^2 = 0, \\ 20\alpha_1 - 8\alpha_4\alpha_2 + 3\alpha_3^2 = 0, \quad (10\alpha_3 - 24\alpha_4^2)\alpha_2 + 9\alpha_4\alpha_3^2 + 500\alpha_0 = 0 \end{array} \right\}.$$

$$D^{(1)} = 64\alpha_1^5 + (352\alpha_4\alpha_2 - 432\alpha_3^2 - 984\alpha_4^2\alpha_3 + 27\alpha_4^4)\alpha_1^4 + ((-984\alpha_3 + 634\alpha_4^2)\alpha_2^2 + (5544\alpha_4\alpha_3^2 - 1800\alpha_4^3\alpha_3$$
$$+108\alpha_4^5 - 2400\alpha_0)\alpha_2 + 972\alpha_3^4 - 3834\alpha_4^2\alpha_3^3 + 2106\alpha_4^4\alpha_3^2 + (-324\alpha_4^6 + 12240\alpha_0\alpha_4)\alpha_3 + 3396\alpha_0\alpha_4^3)\alpha_1^3 + (27\alpha_2^4$$
$$+(-1800\alpha_4\alpha_3 + 368\alpha_4^3)\alpha_2^3 + (-3834\alpha_3^3 + 2592\alpha_4^2\alpha_3^2 - 738\alpha_4^4\alpha_3 + 108\alpha_4^6 - 5220\alpha_0\alpha_4)\alpha_2^2 + (3240\alpha_4\alpha_3^4$$

$$
\begin{aligned}
&-2052\alpha_4^3\alpha_3^3 + (324\alpha_4^5 - 18360\alpha_0)\alpha_3^2 - 13284\alpha_0\alpha_4^2\alpha_3 + 4284\alpha_0\alpha_4^4)\alpha_2 - 729\alpha_3^6 + 486\alpha_4^2\alpha_3^5 - 81\alpha_4^4\alpha_3^4 \\
&+6156\alpha_0\alpha_4\alpha_3^3 + 9288\alpha_0\alpha_4^3\alpha_3^2 + (-7452\alpha_0\alpha_4^5 - 27000\alpha_0^2)\alpha_3 + 1296\alpha_0\alpha_4^7 - 49950\alpha_0^2\alpha_4^2)\alpha_1^2 + (108\alpha_4\alpha_2^5 \\
&+(2106\alpha_3^2 - 738\alpha_4^2\alpha_3 + 8\alpha_4^4)\alpha_2^4 + (-2052\alpha_4\alpha_3^3 + 1296\alpha_4^3\alpha_3^2 - (204\alpha_4^5 - 14580\alpha_0)\alpha_3 - 2124\alpha_0\alpha_4^2)\alpha_2^3 \\
&+(486\alpha_3^5 - 324\alpha_4^2\alpha_3^4 + 54\alpha_4^4\alpha_3^3 + 3024\alpha_0\alpha_4\alpha_3^2 + 1512\alpha_0\alpha_4^3\alpha_3 - 936\alpha_0\alpha_4^5 + 20250\alpha_0^2)\alpha_2^2 + (972\alpha_0\alpha_3^4 \\
&-13608\alpha_0\alpha_4^2\alpha_3^3 + 8316\alpha_0\alpha_4^4\alpha_3^2 + (-1296\alpha_0\alpha_4^6 + 118800\alpha_0^2\alpha_4)\alpha_3 - 26460\alpha_0^2\alpha_4^3)\alpha_2 + 2916\alpha_0\alpha_4\alpha_3^5 \\
&-1944\alpha_0\alpha_4^3\alpha_3^4 + (324\alpha_0\alpha_4^5 + 12150\alpha_0^2)\alpha_3^3 - 86670\alpha_0^2\alpha_4^2\alpha_3^2 + 49572\alpha_0^2\alpha_4^4\alpha_3 - 7776\alpha_0^2\alpha_4^6 + 202500\alpha_0^3\alpha_4)\alpha_1 \\
&+(-324\alpha_3 + 108\alpha_4^2)\alpha_2^6 + (324\alpha_4\alpha_3^2 - 204\alpha_4^3\alpha_3 + 32\alpha_4^5 - 648\alpha_0)\alpha_2^5 + (-81\alpha_3^4 + 54\alpha_4^2\alpha_3^3 - 9\alpha_4^4\alpha_3^2 \\
&-3888\alpha_0\alpha_4\alpha_3 + 1320\alpha_0\alpha_4^3)\alpha_2^4 + (324\alpha_0\alpha_3^3 + 3672\alpha_0\alpha_4^2\alpha_3^2 - 2412\alpha_0\alpha_4^4\alpha_3 + 384\alpha_0\alpha_4^6 - 8100\alpha_0^2\alpha_4)\alpha_2^3 \\
&+(-972\alpha_0\alpha_4\alpha_3^4 + 648\alpha_0\alpha_4^3\alpha_3^3 + (-108\alpha_0\alpha_4^5 - 36450\alpha_0^2)\alpha_3^2 + 12690\alpha_0^2\alpha_4^2\alpha_3 + 108\alpha_0^2\alpha_4^4)\alpha_2^2 + (38880\alpha_0^2\alpha_4\alpha_3^3 \\
&-24624\alpha_0^2\alpha_4^3\alpha_3^2 + (3888\alpha_0^2\alpha_4^5 - 202500\alpha_0^3)\alpha_3 + 40500\alpha_0^3\alpha_4^2)\alpha_2 - 8748\alpha_0^2\alpha_3^5 + 5832\alpha_0^2\alpha_4^2\alpha_3^4 - 972\alpha_0^2\alpha_4^4\alpha_3^3 \\
&+121500\alpha_0^3\alpha_4\alpha_3^2 - 72900\alpha_0^3\alpha_4^3\alpha_3 + 11664\alpha_0^3\alpha_4^5 - 253125\alpha_0^4.
\end{aligned}
$$

$$
\begin{aligned}
E_0^{(1)} &= 256\alpha_1^5 + (-192\alpha_4\alpha_2 - 128\alpha_3^2 + 144\alpha_4^2\alpha_3 - 27\alpha_4^4)\alpha_1^4 + ((144\alpha_3 - 6\alpha_4^2)\alpha_2^2 + (-80\alpha_4\alpha_3^2 + 18\alpha_4^3\alpha_3 \\
&-1600\alpha_0)\alpha_2 + 16\alpha_3^4 - 4\alpha_4^2\alpha_3^3 + 160\alpha_0\alpha_4\alpha_3 - 36\alpha_0\alpha_4^3)\alpha_1^3 + (-27\alpha_2^4 + (18\alpha_4\alpha_3 - 4\alpha_4^3)\alpha_2^3 + (-4\alpha_3^3 + \alpha_4^2\alpha_3^2 \\
&+1020\alpha_0\alpha_4)\alpha_2^2 + (560\alpha_0\alpha_3^2 - 746\alpha_0\alpha_4^2\alpha_3 + 144\alpha_0\alpha_4^4)\alpha_2 + 24\alpha_0\alpha_4\alpha_3^3 - 6\alpha_0\alpha_4^3\alpha_3^2 + 2000\alpha_0^2\alpha_3 - 50\alpha_0^2\alpha_4^2)\alpha_1^2 \\
&+((-630\alpha_0\alpha_3 + 24\alpha_0\alpha_4^2)\alpha_2^3 + (356\alpha_0\alpha_4\alpha_3^2 - 80\alpha_0\alpha_4^3\alpha_3 + 2250\alpha_0^2)\alpha_2^2 + (-72\alpha_0\alpha_3^4 + 18\alpha_0\alpha_4^2\alpha_3^3 - 2050\alpha_0^2\alpha_4\alpha_3 \\
&+160\alpha_0^2\alpha_4^3)\alpha_2 - 900\alpha_0^2\alpha_3^3 + 1020\alpha_0^2\alpha_4^2\alpha_3^2 - 192\alpha_0^2\alpha_4^4\alpha_3 - 2500\alpha_0^3\alpha_4)\alpha_1 + 108\alpha_0\alpha_2^5 + (-72\alpha_0\alpha_4\alpha_3 \\
&+16\alpha_0\alpha_4^3)\alpha_2^4 + (16\alpha_0\alpha_3^3 - 4\alpha_0\alpha_4^2\alpha_3^2 - 900\alpha_0^2\alpha_4)\alpha_2^3 + (825\alpha_0^2\alpha_3^2 + 560\alpha_0^2\alpha_4^2\alpha_3 - 128\alpha_0^2\alpha_4^4)\alpha_2^2 + (-630\alpha_0^2\alpha_4\alpha_3^3 \\
&+144\alpha_0^2\alpha_4^3\alpha_3^2 - 3750\alpha_0^3\alpha_3 + 2000\alpha_0^3\alpha_4^2)\alpha_2 + 108\alpha_0^2\alpha_3^5 - 27\alpha_0^2\alpha_4^2\alpha_3^4 + 2250\alpha_0^3\alpha_4\alpha_3^2 - 1600\alpha_0^3\alpha_4^3\alpha_3 + 256\alpha_0^3\alpha_4^5 \\
&+3125\alpha_0^4.
\end{aligned}
$$

The details of the number of preimages are shown in the following Table 2, where $E_k^{(1)}$ $(k = 0, \cdots, 4)$ mean the coefficients of equation

$$8503056r^5 - 196835E_4^{(1)}r^4 + 11664E_3^{(1)}r^3 - 864E_2^{(1)}r^2 + 256E_1^{(1)}E_0^{(1)}r - 256(E_0^{(1)})^2 = 0$$

obtained by eliminating b_1, b_0, a_1, a_0, c_1 from the resultant $r = \mathrm{Resul}_z(\widehat{P}, Q)$, and $I_k^{(1)}$ $(k = 2, 3)$ are given as follows.

$$
\begin{aligned}
I_2^{(1)} = \{&64\alpha_1^5 + (352\alpha_4\alpha_2 - 432\alpha_3^2 - 984\alpha_4^2\alpha_3 + 27\alpha_4^4)\alpha_1^4 + ((-984\alpha_3 + 634\alpha_4^2)\alpha_2^2 + (5544\alpha_4\alpha_3^2 \\
&- 1800\alpha_4^3\alpha_3 + 108\alpha_4^5 - 2400\alpha_0)\alpha_2 + 972\alpha_3^4 - 3834\alpha_4^2\alpha_3^3 + 2106\alpha_4^4\alpha_3^2 - (324\alpha_4^6 - 12240\alpha_0\alpha_4)\alpha_3 \\
&+ 3396\alpha_0\alpha_4^3)\alpha_1^3 + (27\alpha_2^4 + (-1800\alpha_4\alpha_3 + 368\alpha_4^3)\alpha_2^3 + (-3834\alpha_3^3 + 2592\alpha_4^2\alpha_3^2 - 738\alpha_4^4\alpha_3 \\
&+ 108\alpha_4^6 - 5220\alpha_0\alpha_4)\alpha_2^2 + (3240\alpha_4\alpha_3^4 - 2052\alpha_4^3\alpha_3^3 + (324\alpha_4^5 - 18360\alpha_0)\alpha_3^2 - 13284\alpha_0\alpha_4^2\alpha_3 \\
&+ 4284\alpha_0\alpha_4^4)\alpha_2 - 729\alpha_3^6 + 486\alpha_4^2\alpha_3^5 - 81\alpha_4^4\alpha_3^4 + 6156\alpha_0\alpha_4\alpha_3^3 + 9288\alpha_0\alpha_4^3\alpha_3^2 \\
&+ (-7452\alpha_0\alpha_4^5 - 27000\alpha_0^2)\alpha_3 + 1296\alpha_0\alpha_4^7 - 49950\alpha_0^2\alpha_4^2)\alpha_1^2 + (108\alpha_4\alpha_2^5 + (2106\alpha_3^2 - 738\alpha_4^2\alpha_3 \\
&+ 8\alpha_4^4)\alpha_2^4 + (-2052\alpha_4\alpha_3^3 + 1296\alpha_4^3\alpha_3^2 + (-204\alpha_4^5 + 14580\alpha_0)\alpha_3 - 2124\alpha_0\alpha_4^2)\alpha_2^3 \\
&+ (486\alpha_3^5 - 324\alpha_4^2\alpha_3^4 + 54\alpha_4^4\alpha_3^3 + 3024\alpha_0\alpha_4\alpha_3^2 + 1512\alpha_0\alpha_4^3\alpha_3 - 936\alpha_0\alpha_4^5 + 20250\alpha_0^2)\alpha_2^2 \\
&+ (972\alpha_0\alpha_3^4 - 13608\alpha_0\alpha_4^2\alpha_3^3 + 8316\alpha_0\alpha_4^4\alpha_3^2 + (-1296\alpha_0\alpha_4^6 + 118800\alpha_0^2\alpha_4)\alpha_3 - 26460\alpha_0^2\alpha_4^3)\alpha_2 \\
&+ 2916\alpha_0\alpha_4\alpha_3^5 - 1944\alpha_0\alpha_4^3\alpha_3^4 + (324\alpha_0\alpha_4^5 + 12150\alpha_0^2)\alpha_3^3 - 86670\alpha_0^2\alpha_4^2\alpha_3^2 + 49572\alpha_0^2\alpha_4^4\alpha_3 \\
&- 7776\alpha_0^2\alpha_4^6 + 202500\alpha_0^3\alpha_4)\alpha_1 - (324\alpha_3 - 108\alpha_4^2)\alpha_2^6 + (324\alpha_4\alpha_3^2 - 204\alpha_4^3\alpha_3 + 32\alpha_4^5 - 648\alpha_0)\alpha_2^5 \\
&+ (-81\alpha_3^4 + 54\alpha_4^2\alpha_3^3 - 9\alpha_4^4\alpha_3^2 - 3888\alpha_0\alpha_4\alpha_3 + 1320\alpha_0\alpha_4^3)\alpha_2^4 + (324\alpha_0\alpha_3^3 + 3672\alpha_0\alpha_4^2\alpha_3^2 \\
&- 2412\alpha_0\alpha_4^4\alpha_3 + 384\alpha_0\alpha_4^6 - 8100\alpha_0^2\alpha_4)\alpha_2^3 + (-972\alpha_0\alpha_4\alpha_3^4 + 648\alpha_0\alpha_4^3\alpha_3^3 + (-108\alpha_0\alpha_4^5 \\
&- 36450\alpha_0^2)\alpha_3^2 + 12690\alpha_0^2\alpha_4^2\alpha_3 + 108\alpha_0^2\alpha_4^4)\alpha_2^2 + (38880\alpha_0^2\alpha_4\alpha_3^3 - 24624\alpha_0^2\alpha_4^3\alpha_3^2 + (3888\alpha_0^2\alpha_4^5 \\
&- 202500\alpha_0^3)\alpha_3 + 40500\alpha_0^3\alpha_4^2)\alpha_2 - 8748\alpha_0^2\alpha_3^5 + 5832\alpha_0^2\alpha_4^2\alpha_3^4 - 972\alpha_0^2\alpha_4^4\alpha_3^3 + 121500\alpha_0^3\alpha_4\alpha_3^2 \\
&- 72900\alpha_0^3\alpha_4^3\alpha_3 + 11664\alpha_0^3\alpha_4^5 - 253125\alpha_0^4 = 0\} \setminus \{E_0^{(1)} = E_2^{(1)} = E_3^{(1)} = 0\}.
\end{aligned}
$$

$I_3^{(1)} = \{p_0 = p_1 = p_2 = p_3 = 0\} \setminus \left(I_2^{(1)} \cup \{E_0^{(1)} = E_2^{(1)} = 0\}\right)$, where

$$\begin{aligned}
p_0 =& 500\alpha_1^3 + (-600\alpha_4\alpha_2 + 675\alpha_3^2 - 360\alpha_4^2\alpha_3 + 72\alpha_4^4)\alpha_1^2 + ((4050\alpha_3 - 1380\alpha_4^2)\alpha_2^2 \\
&+ (-5400\alpha_4\alpha_3^2 + 3528\alpha_4^3\alpha_3 - 576\alpha_4^5)\alpha_2 - 2430\alpha_3^4 + 5508\alpha_4^2\alpha_3^3 - 3834\alpha_4^4\alpha_3^2 + 1080\alpha_4^6\alpha_3 - 108\alpha_4^8)\alpha_1 \\
&+ 675\alpha_2^4 + (-3240\alpha_4\alpha_3 + 1048\alpha_4^3)\alpha_2^3 + (-7290\alpha_3^3 + 12744\alpha_4^2\alpha_3^2 - 5922\alpha_4^4\alpha_3 + 828\alpha_4^6)\alpha_2^2 \\
&+ (9720\alpha_4\alpha_3^4 - 16200\alpha_4^3\alpha_3^3 + 9504\alpha_4^5\alpha_3^2 - 2376\alpha_4^7\alpha_3 + 216\alpha_4^9)\alpha_2 - 3645\alpha_3^6 \\
&+ 5832\alpha_4^2\alpha_3^5 - 3402\alpha_4^4\alpha_3^4 + 864\alpha_4^6\alpha_3^3 - 81\alpha_4^8\alpha_3^2, \\
p_1 =& 20\alpha_1^2 + (74\alpha_4\alpha_2 - 9\alpha_3^2 - 42\alpha_4^2\alpha_3 + 12\alpha_4^4)\alpha_1 + (-207\alpha_3 + 68\alpha_4^2)\alpha_2^2 + (252\alpha_4\alpha_3^2 - 156\alpha_4^3\alpha_3 + 24\alpha_4^5 \\
&- 450\alpha_0)\alpha_2 - 81\alpha_3^4 + 54\alpha_4^2\alpha_3^3 - 9\alpha_4^4\alpha_3^2 + 270\alpha_0\alpha_4\alpha_3 - 72\alpha_0\alpha_4^3, \\
p_2 =& 490\alpha_4\alpha_1^2 + ((-1845\alpha_3 + 616\alpha_4^2)\alpha_2 + 1899\alpha_4\alpha_3^2 - 1398\alpha_4^3\alpha_3 + 258\alpha_4^5 - 2250\alpha_0)\alpha_1 - 135\alpha_2^3 \\
&+ (567\alpha_4\alpha_3 - 188\alpha_4^3)\alpha_2^2 + (-405\alpha_3^3 + 18\alpha_4^2\alpha_3^2 + 111\alpha_4^4\alpha_3 - 24\alpha_4^6)\alpha_2 + 81\alpha_4\alpha_3^4 \\
&- 54\alpha_4^3\alpha_3^3 + (9\alpha_4^5 - 4050\alpha_0)\alpha_3^2 + 3510\alpha_0\alpha_4^2\alpha_3 - 738\alpha_0\alpha_4^4, \\
p_3 =& (1600\alpha_3 - 1865\alpha_4^2)\alpha_1^2 + (75\alpha_2^2 + (7855\alpha_4\alpha_3 - 2624\alpha_4^3)\alpha_2 - 2340\alpha_3^3 - 6321\alpha_4^2\alpha_3^2 + 5502\alpha_4^4\alpha_3 \\
&- 1062\alpha_4^6)\alpha_1 + 645\alpha_4\alpha_2^3 + (-6120\alpha_3^2 + 1327\alpha_4^2\alpha_3 + 232\alpha_4^4)\alpha_2^2 + (10305\alpha_4\alpha_3^3 - 7962\alpha_4^3\alpha_3^2 \\
&+ 1941\alpha_4^5\alpha_3 - 144\alpha_4^7)\alpha_2 - 2835\alpha_3^5 + 2376\alpha_4^2\alpha_3^4 - 639\alpha_4^4\alpha_3^3 + (54\alpha_4^6 + 20250\alpha_0\alpha_4)\alpha_3^2 \\
&- 16650\alpha_0\alpha_4^3\alpha_3 + 3402\alpha_0\alpha_4^5 + 28125\alpha_0^2.
\end{aligned}$$

$\#(\Phi_4^{(1)})^{-1}(\alpha)$	α	Remark
0	$E^{(1)}(4)$	
1	$E_0^{(1)} = E_2^{(1)} = E_3^{(1)} = 0,\ E_4^{(1)} \neq 0$	inv=1
2	$E_0^{(1)} = E_2^{(1)} = 0,\ E_3^{(1)} \neq 0$	inv=2
3	$E_0^{(1)} = 0,\ E_2^{(1)} \neq 0$	inv=3
4	$\emptyset$	inv=4
1	$\emptyset$	5-ple
2	$\emptyset$	4-ple
2	$\emptyset$	double+triple
3	$I_3^{(1)}$	triple
3	$\emptyset$	double+double
4	$I_2^{(1)}$	double
5	otherwise	

Table 2: The number of inverse images.

Next, if ∞ is a double critical point, we have the following result.

Proposition 9
The ramification locus of $\Phi_4^{(2)}$ is given by $c_1 + 3b_0^2 - 2c_2b_0 = 0$. $\Phi_4^{(2)}(CB_4^{(2)}) = \mathbb{C}^4 - E^{(2)}(4)$, and $\Phi_4^{(2)}$ is 3-valent on the set of points in $\mathbb{C}^4 - E^{(2)}(4)$ satisfying

$$\begin{aligned}
D^{(2)} =& 108\alpha_1^2 + (-108\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_2^3 - 9\alpha_3^2\alpha_2^2 \neq 0, \\
E_0^{(2)} =& 27\alpha_0^2\alpha_3^4 + (4\alpha_1^3 - 18\alpha_0\alpha_2\alpha_1)\alpha_3^3 + ((-\alpha_2^2 + 6\alpha_0)\alpha_1^2 + 4\alpha_0\alpha_2^3 - 144\alpha_0^2\alpha_2)\alpha_3^2 \\
&+ (-18\alpha_2\alpha_1^3 + (80\alpha_0\alpha_2^2 + 192\alpha_0^2)\alpha_1)\alpha_3 + 27\alpha_1^4 + (4\alpha_2^3 - 144\alpha_0\alpha_2)\alpha_1^2 \\
&- 16\alpha_0\alpha_2^4 + 128\alpha_0^2\alpha_2^2 - 256\alpha_0^3 \neq 0.
\end{aligned}$$

Moreover, the defining equation of $E^{(2)}(4)$ is the algebraic variety defined by

$$\{3\alpha_3^2 - 8\alpha_2 = 0, \quad \alpha_3^3 - 16\alpha_1 = 0, \quad \alpha_3^4 - 256\alpha_0 = 0\}.$$

The details of the number of preimages are shown in the following Table 3, where $E_k^{(2)}$ $(k = 0, \cdots, 2)$ mean the coefficients of equation

$$256r^3 - 3E_2^{(2)}r^2 + 18E_1^{(2)}r - 27E_0^{(2)} = 0$$

obtained by eliminating b_0, a_0, c_2, c_1 from the resultant $r = \mathrm{Resul}_z(\widehat{P}, Q)$, and

$$I_2^{(2)} = \{108\alpha_1^2 + (-108\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_2^3 - 9\alpha_3^2\alpha_2^2 = 0\} \setminus \{E_0^{(2)} = E_1^{(2)} = 0\}.$$
$$I_3^{(2)} = \{8\alpha_2 - 3\alpha_3^2 = 8\alpha_1 - 4\alpha_3\alpha_2 + \alpha_3^3 = 0\} \setminus \{E_0^{(2)} = 0\}.$$

$\#(\Phi_4^{(2)})^{-1}(\alpha)$	α	Remark
0	$E^{(2)}(4)$	$(*)$
1	$E_0^{(2)} = E_1^{(2)} = 0, E_2^{(2)} \neq 0$	inv=1
2	$E_0^{(2)} = 0,\ E_1^{(2)} \neq 0$	inv=2
1	$I_3^{(2)}$	triple
2	$I_2^{(2)}$	double
3	otherwise	

Table 3: The number of inverse images.

Finally, since the map $\Phi_4^{(3)} : CB_4^{(3)} \to \mathbb{C}^3$ is clearly bijective, we have obtained complete description for the case that $d = 4$.

For $d = 3,\ 4$, the complete answer for the problem of Goldberg was obtained.

3 Homogenized Bell family

In the previous section, we see that the generalized Bell locus $CB_d^{(k)}$ gives good coordinate system for the space $X_d^{(k)}$ of equivalence classes, for each $k = 0, \cdots, d-1$. In this section, we introduce another family of rational maps that gives "coordinate system" without depending on the multiplicity of critical points at ∞.

Considering the composition of

$$F(z) = z^{k+1} + c_k z^k + \cdots + c_1 z + \frac{a_{d-k-2}z^{d-k-2} + \cdots + a_0}{z^{d-k-1} + b_{d-k-2}z^{d-k-2} + \cdots + b_0} \in CB_d^{(k)},$$

and linear translation $M(z) = z - \beta$, we have

$$M \circ F(z) = \frac{z^d + (b_{d-k-2} + c_k)z^{d-1} + \cdots + \tilde{a}_{d-k-2}z^{d-k-2} + \tilde{a}_{d-k}z^{d-k} + \cdots + a_0}{z^{d-k-1} + \cdots + b_0},$$

where

$$\beta = \sum_{j=0}^{M} c_j b_{d-k-1-j} \qquad (c_{k+1} = 1,\ M = \min\{k+1, d-k-1\}).$$

Therefore, for $k = 0, \cdots, d-1$, we can use the following family

$$MB_d^{(k)} := \left\{ \frac{z^d + a_{d-1}z^{d-1} + \cdots + a_{d-k-2}z^{d-k-2} + a_{d-k}z^{d-k} + \cdots + a_0}{z^{d-k-1} + \cdots + b_0} \right\},$$

instead of the generalized Bell locus $CB_d^{(k)}$.

Let HB_d be the family of rational maps of degree d consisting of all P/Q, for

$$\begin{aligned} P(z) &= z^d + (1 - b_{d-1})a_{d-1}z^{d-1} + (1 - (1 - b_{d-1})b_{d-2})a_{d-2}z^{d-2} + \cdots \\ &\quad \cdots + (1 - (1 - b_{d-1}) \cdots (1 - b_1)b_0)a_0, \\ Q(z) &= b_{d-1}z^{d-1} + \cdots + b_0, \end{aligned}$$

with $\mathrm{Resul}_z(P, Q) \neq 0$, where coefficient parameters are given as elements of projective spaces, $(b_{d-1} : \cdots : b_0) \in \mathbb{P}^{d-1}(\mathbb{C})$ and $(1 : a_{d-1} : \cdots : a_0) \in \mathbb{P}^d(\mathbb{C})$.

Moreover, we define $HB_d^{(k)}$ $(k = 0, \cdots, d-1)$ are the classes of rational maps with k-ple critical point at ∞, i.e.,

$$HB_d^{(k)} = \left\{ \frac{P}{Q} : \begin{array}{l} P(z) = z^d + (1 - b_{d-1})a_{d-1}z^{d-1} + \cdots + (1 - (1 - b_{d-1}) \cdots (1 - b_1)b_0)a_0, \\ Q(z) = z^{d-k-1} + b_{d-k-2}z^{d-k-2} + \cdots + b_0, \quad \text{with } \mathrm{Reul}_z(P, Q) \neq 0 \end{array} \right\}.$$

Remark 3
For each k, the coefficient a_{d-k-1} of each rational map in $HB_d^{(k)}$ is vanished. Therefore, we have

$$HB_d^{(k)} \cong \{(b_{d-k-2}, \cdots, b_0, a_{d-1}, \cdots, a_{d-k-2}, a_{d-k}, \cdots, a_0) \in \mathbb{C}^{2d-2-k} : \mathrm{Resul}_z(P, Q) \neq 0\}$$

Moreover, HB_d is the disjoint union of $HB_d^{(0)}, \cdots, HB_d^{(d-1)}$.

From the above argument, we have

Theorem 10
For every $R \in HB_d^{(k)}$, $[R]$ belongs to $X_d^{(k)}$ for every k, and for each element $[R]$ in $X_d^{(k)}$, there is a unique R' in $HB_d^{(k)}$ with $[R'] = [R]$.

Hence, for each locus $X_d^{(k)}$ has a system of coordinates consisting of coefficients of representatives R in $HB_d^{(k)}$.

Here, we consider the map $\widehat{\Phi}_d$ of HB_d to $\mathbb{P}^{2d-2}(\mathbb{C})$ defined from the equation

$$P'_{(\boldsymbol{b},\boldsymbol{a})}(z)Q_{(\boldsymbol{b},\boldsymbol{a})}(z) - P_{(\boldsymbol{b},\boldsymbol{a})}(z)Q'_{(\boldsymbol{b},\boldsymbol{a})}(z) = \alpha_{2d-2}z^{2d-2} + \alpha_{2d-3}z^{2d-3} + \cdots + \alpha_1 z + \alpha_0,$$

by sending

$$(\boldsymbol{b}, \boldsymbol{a}) = \Big((\underbrace{0 : \cdots : 0}_{k} : 1 : b_{d-k-2} : \cdots : b_0), (1 : a_{d-1} : \cdots : a_{d-k-2} : 0 : a_{d-k} : \cdots : a_0)\Big)$$

to

$$\boldsymbol{\alpha} = (\underbrace{0 : \cdots : 0}_{k} : k+1 : \alpha_{2d-k-3} : \cdots : \alpha_0) \in \mathbb{P}^{2d-2}(\mathbb{C}).$$

3.1 The case of degree 3

Recall that a rational map in HB_3 has following form,

$$R(z) = \frac{z^3 + (1-b_2)a_2 z^2 + (1-(1-b_2)b_1)a_1 z + (1-(1-b_2)(1-b_1)b_0)a_0}{b_2 z^2 + b_1 z + b_0}.$$

Theorem 11
$\widehat{\Phi}_3(HB_3) = \mathbb{P}^4(\mathbb{C}) - \widehat{E}(3)$ and $\Phi_3(HB_3)$ is 2-valent on the the set of the points in $\mathbb{P}^4(\mathbb{C}) - \widehat{E}(3)$ satisfying that

$$\mathrm{Discr}(q) \neq 0, \quad \widehat{E}_0(\boldsymbol{\alpha}) \neq 0,$$

where,

$$\begin{aligned}
\mathrm{Discr}(q) =& 3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0\alpha_4, \\
\widehat{E}_0(\boldsymbol{\alpha}) =& 27\alpha_4^2\alpha_1^4 + (-18\alpha_4\alpha_3\alpha_2 + 4\alpha_3^3)\alpha_1^3 + (4\alpha_4\alpha_2^3 - \alpha_3^2\alpha_2^2 - 144\alpha_0\alpha_4^2\alpha_2 + 6\alpha_0\alpha_4\alpha_3^2)\alpha_1^2 \\
&+ (80\alpha_0\alpha_4\alpha_3\alpha_2^2 - 18\alpha_0\alpha_3^3\alpha_2 + 192\alpha_0^2\alpha_4^2\alpha_3)\alpha_1 - 16\alpha_0\alpha_4\alpha_2^4 + 4\alpha_0\alpha_3^2\alpha_2^3 \\
&+ 128\alpha_0^2\alpha_4^2\alpha_2^2 - 144\alpha_0^2\alpha_4\alpha_3^2\alpha_2 + 27\alpha_0^2\alpha_3^4 - 256\alpha_0^3\alpha_4^3 = 0.
\end{aligned}$$

Here, the exceptional locus $\widehat{E}(3)$ is the algebraic variety defined by

$$\begin{aligned}
\widehat{E}(3) = \Big\{& 108\alpha_4^2\alpha_1^2 + (-108\alpha_4\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_4\alpha_2^3 - 9\alpha_3^2\alpha_2^2 = 0, \\
& 3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0\alpha_4 = 0, \quad -27\alpha_4\alpha_1^2 + 27\alpha_3\alpha_2\alpha_1 - 8\alpha_2^3 - 27\alpha_0\alpha_3^2 = 0\Big\}.
\end{aligned} \tag{4}$$

Proof The map $\widehat{\Phi}_3$ is defined by

$$(\boldsymbol{b}, \boldsymbol{a}) = ((b_2 : b_1 : b_0), (1 : a_2 : a_1 : a_0)) \mapsto \boldsymbol{\alpha} = (\alpha_4 : \cdots : \alpha_0),$$

where

$$\begin{aligned}
\alpha_4 &= b_2, \\
\alpha_3 &= 2b_1, \\
\alpha_2 &= (-a_1 b_2^2 + (-a_2 + a_1)b_2 + a_2)b_1 - a_1 b_2 + 3b_0, \\
\alpha_1 &= (2a_0 b_0 b_2^2 - 2a_0 b_0 b_2)b_1 - 2a_0 b_0 b_2^2 + (-2b_0 a_2 + 2a_0 b_0 - 2a_0)b_2 + 2b_0 a_2, \\
\alpha_0 &= (a_0 b_0 b_2 - a_0 b_0)b_1^2 + ((b_0 a_1 - a_0 b_0)b_2 - b_0 a_1 + a_0 b_0 - a_0)b_1 + b_0 a_1.
\end{aligned} \tag{5}$$

The map $\widehat{\Phi}_3$ is not defined if and only if the coefficients of R satisfy the following condition,

$$\begin{aligned}
r =& \mathrm{Resul}_z(\mathrm{nm}\, R, \mathrm{dn}\, R) \\
=& (a_0 b_0 b_2 - a_0 b_0)b_1^4 + (a_0 b_0 a_1 b_2^4 + (a_0 b_0 a_2 - 2a_0 b_0 a_1)b_2^3 + (-2a_0 b_0 a_2 + a_0 b_0 a_1)b_2^2 \\
&+ (a_0 b_0 a_2 + b_0 a_1 - a_0 b_0)b_2 - b_0 a_1 + a_0 b_0 - a_0)b_1^3 + (a_0^2 b_0^2 b_2^5 + (b_0 a_1^2 - a_0 b_0 a_1 - 2a_0^2 b_0^2)b_2^4 \\
&+ ((b_0 a_1 - a_0 b_0)a_2 - 2b_0 a_1^2 + (3a_0 b_0 - a_0)a_1 + a_0^2 b_0^2)b_2^3 + ((-2b_0 a_1 + 2a_0 b_0 - a_0)a_2 \\
&+ b_0 a_1^2 + (-2a_0 b_0 + a_0)a_1 - 3a_0 b_0^2)b_2^2 + ((b_0 a_1 - a_0 b_0 + a_0)a_2 + 3a_0 b_0^2)b_2 + b_0 a_1)b_1^2 \\
&+ (-2a_0^2 b_0^2 b_2^5 + (-2a_0 b_0^2 a_2 + 4a_0^2 b_0^2 - 2a_0^2 b_0)b_2^4 + (4a_0 b_0^2 a_2 + 2b_0 a_1^2 - a_0 b_0 a_1 \\
&- 2a_0^2 b_0^2 + 2a_0^2 b_0)b_2^3 + ((b_0 a_1 - 2a_0 b_0^2)a_2 - 2b_0 a_1^2 + (-2b_0^2 + a_0 b_0 - a_0)a_1 + 3a_0 b_0^2)b_2^2 \\
&+ ((-b_0 a_1 + b_0^2)a_2 + 2b_0^2 a_1 - 3a_0 b_0^2 + 3a_0 b_0)b_2 - b_0^2 a_2)b_1 + a_0^2 b_0^2 b_2^5 \\
&+ (2a_0 b_0^2 a_2 - 2a_0^2 b_0^2 + 2a_0^2 b_0)b_2^4 + (b_0^2 a_2^2 + (-4a_0 b_0^2 + 2a_0 b_0)a_2 + a_0^2 b_0^2 - 2a_0^2 b_0 + a_0^2)b_2^3 \\
&+ (-2b_0^2 a_2^2 + (2a_0 b_0^2 - 2a_0 b_0)a_2 + b_0 a_1^2)b_2^2 + (b_0^2 a_2^2 - 2b_0^2 a_1)b_2 + b_0^3 = 0.
\end{aligned} \tag{6}$$

For each $\boldsymbol{\alpha} \in \mathbb{P}^4(\mathbb{C}) - \widehat{E}(3)$, corresponding coefficients b_2, b_1, b_0 are determined as solution of

$$\begin{aligned} q(b_0) &= 12b_0^2 - 4\alpha_2 b_0 + \alpha_3\alpha_1 - 4\alpha_0\alpha_4 = 0, \\ b_2 &= \alpha_4, \\ b_1 &= \frac{1}{2}\alpha_3. \end{aligned}$$

And we can also check that the other coefficients a_0, a_1, a_2 are uniquely determined by $\boldsymbol{\alpha}$ and $(b_2 : b_1 : b_0)$.

Therefore, $\#\widehat{\Phi}_3(\boldsymbol{\alpha})^{-1} = 2$ except for

$$\mathrm{Discr}(q) = 3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0\alpha_4 = 0. \tag{7}$$

Here, eliminating six variables $b_2, b_1, b_0, a_2, a_1, a_0$ from the expression (6) by using (5), we have the following equation

$$\begin{aligned} &432r^2 + (216\alpha_4\alpha_1^2 - 72\alpha_3\alpha_2\alpha_1 + 16\alpha_2^3 - 576\alpha_0\alpha_4\alpha_2 + 216\alpha_0\alpha_3^2)r \\ &+ 27\alpha_4^2\alpha_1^4 + (-18\alpha_4\alpha_3\alpha_2 + 4\alpha_3^3)\alpha_1^3 + (4\alpha_4\alpha_2^3 - \alpha_3^2\alpha_2^2 - 144\alpha_0\alpha_4^2\alpha_2 + 6\alpha_0\alpha_4\alpha_3^2)\alpha_1^2 \\ &+ (80\alpha_0\alpha_4\alpha_3\alpha_2^2 - 18\alpha_0\alpha_3^3\alpha_2 + 192\alpha_0^2\alpha_4^2\alpha_3)\alpha_1 - 16\alpha_0\alpha_4\alpha_2^4 + 4\alpha_0\alpha_3^2\alpha_2^3 \\ &+ 128\alpha_0^2\alpha_4^2\alpha_2^2 - 144\alpha_0^2\alpha_4\alpha_3^2\alpha_2 + 27\alpha_0^2\alpha_3^4 - 256\alpha_0^3\alpha_4^3 = 0. \end{aligned} \tag{8}$$

Here, the exceptional locus $\widehat{E}(3)$ corresponds to the condition that this equation has 0 as unique solution. Therefore a defining equation of the exceptional locus $\widehat{E}(3)$ is given by

$$\begin{aligned} \widehat{E}(3) = \Big\{&27\alpha_4\alpha_1^2 - 9\alpha_3\alpha_2\alpha_1 + 2\alpha_2^3 - 72\alpha_0\alpha_4\alpha_2 + 27\alpha_0\alpha_3^2 = 0, \\ &27\alpha_4^2\alpha_1^4 + (-18\alpha_4\alpha_3\alpha_2 + 4\alpha_3^3)\alpha_1^3 + (4\alpha_4\alpha_2^3 - \alpha_3^2\alpha_2^2 - 144\alpha_0\alpha_4^2\alpha_2 + 6\alpha_0\alpha_4\alpha_3^2)\alpha_1^2 \\ &+ (80\alpha_0\alpha_4\alpha_3\alpha_2^2 - 18\alpha_0\alpha_3^3\alpha_2 + 192\alpha_0^2\alpha_4^2\alpha_3)\alpha_1 - 16\alpha_0\alpha_4\alpha_2^4 + 4\alpha_0\alpha_3^2\alpha_2^3 \\ &+ 128\alpha_0^2\alpha_4^2\alpha_2^2 - 144\alpha_0^2\alpha_4\alpha_3^2\alpha_2 + 27\alpha_0^2\alpha_3^4 - 256\alpha_0^3\alpha_4^3 = 0\Big\}, \end{aligned}$$

and can be simplified as (4).

Let $\widehat{E}_0(\boldsymbol{\alpha})$ be the constant term of (8). The locus $\widehat{E}_0(\boldsymbol{\alpha}) = 0$ corresponds to the condition that the equation (8) has 0 as one of solutions. Then, $\#\widehat{\Phi}(\boldsymbol{\alpha}) < 2$ on the locus $\widehat{E}_0(\boldsymbol{\alpha}) = 0$.

Moreover we can check that the equation

$$\alpha_4 z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$$

has a solution of multiplicity at least 3 if and only if $\boldsymbol{\alpha}$ belongs to $\widehat{E}(3)$. ∎

Now, we investigate in detail about the structure of the map $\widehat{\Phi}_3$.

- **On the affine 4-space U_4:**

  On the space, $U_4 = \{(1 : \alpha_3 : \alpha_2 : \alpha_1 : \alpha_0)\} \cong \mathbb{C}^4 \subset \mathbb{P}^4(\mathbb{C})$, the ramification locus (7) and the degeneration locus $\widehat{E}_0(\boldsymbol{\alpha}) = 0$ are written by

  $$3\alpha_3\alpha_1 - \alpha_2^2 - 12\alpha_0 = 0 \quad \text{and} \quad \widehat{E}_0(1, \alpha_3, \cdots, \alpha_0) = 0,$$

  respectively. Moreover, the exceptional locus is written by

  $$\widehat{E}(3) \cup U_4 = \{108\alpha_1^2 + (-108\alpha_3\alpha_2 + 27\alpha_3^3)\alpha_1 + 32\alpha_2^3 - 9\alpha_3^2\alpha_2^2 = 0, \quad -3\alpha_3\alpha_1 + \alpha_2^2 + 12\alpha_0 = 0\}.$$

  and we can check that this algebraic variety coincides with the algebraic variety $E^{(0)}(3)$ in Proposition 5.

- **On the hyperplane $H_3 = \{(0 : \alpha_3 : \cdots : \alpha_0)\} \cong \mathbb{P}^3(\mathbb{C})$:**

- **On the affine 3-space U_3:**
  On the space $U_3 = \{(0:1:\alpha_2:\alpha_1:\alpha_0)\} \cong \mathbb{C}^3$, the ramification locus (7), the degeneration locus $\widehat{E}_0(\boldsymbol{\alpha}) = 0$ and the exceptional locus are written by

$$\begin{aligned} &3\alpha_1 - \alpha_2^2 = 0, \\ &\widehat{E}_0(0,1,\alpha_2,\alpha_1,\alpha_0) = 4\alpha_1^3 - \alpha_2^2\alpha_1^2 - 18\alpha_0\alpha_2\alpha_1 + 4\alpha_0\alpha_2^3 + 27\alpha_0^2 = 0, \end{aligned}$$

  and

$$\begin{aligned} \widehat{E}(3) \cup U_3 &= \{3\alpha_1 - \alpha_2^2 = 0,\ 9\alpha_2\alpha_1 - 2\alpha_2^3 - 27\alpha_0 = 0\} \\ &= \{3\alpha_1 - \alpha_2^2 = 0,\ \alpha_2^3 - 27\alpha_0 = 0\}, \end{aligned}$$

  respectively. And the last algebraic variety is coincides with the algebraic variety $E^{(1)}(3)$ in Proposition 6.
- **On the hyperplane $H_2 = \{(0:0:\alpha_2:\alpha_1:\alpha_0)\} \cong \mathbb{P}^2(\mathbb{C})$:**
  * **On the affine 2-space U_2:**
    On the space $U_2 = \{(0:0:1:\alpha_1:\alpha_0)\} \cong \mathbb{C}^2$, $\mathrm{Discr}(q) = -1$, $\widehat{E}_0(\boldsymbol{\alpha}) \equiv 0$ and

$$\widehat{E}(3) \cup U_2 = \emptyset.$$

    This fact is coincides with the result that $\Phi_3^{(2)}$ is bijective.
  * **On the hyperplane $H_1 = \{(0:0:0:\alpha_1:\alpha_0)\} \cong \mathbb{P}^1(\mathbb{C})$:**
    On the hyperplane H_1, each rational map is non-admissible, because

$$\widehat{E}(3) \supset H_1.$$

The above result can be summarized as following Tables 4 and 5.

<table>
<tr><td rowspan="4">The space of critical sets
$\mathbb{P}^4(\mathbb{C}) \supset \{\boldsymbol{\alpha}\}$</td><td colspan="4">The affine space $U_4 \supset \left\{\begin{array}{c}\text{the critical sets of all rational}\\ \text{functions that } \infty \text{ is non-critical}\end{array}\right\}$</td></tr>
<tr><td rowspan="3">$H_3 = \mathbb{P}^3(\mathbb{C})$</td><td colspan="3">$U_3 \supset \{$"$\infty$ is simple critical"$\}$</td></tr>
<tr><td rowspan="2">$H_2 = \mathbb{P}^2(\mathbb{C})$</td><td colspan="2">$U_2 \supset \{$"$\infty$ is double critical"$\}$</td></tr>
<tr><td>$H_1 = \mathbb{P}^1(\mathbb{C})$</td><td>non-admissible</td></tr>
</table>

Table 4: Construction of the map $\widehat{\Phi}_3$

$\#(\widehat{\Phi}_3)^{-1}(\boldsymbol{\alpha})$	$\boldsymbol{\alpha}$	Remark
0	$\widehat{E}(3)$	
1	$\{\widehat{E}_0(\boldsymbol{\alpha}) = 0\} - \widehat{E}(3)$	inv=1
1	$\{\mathrm{Discr}(q) = 0\} - \widehat{E}(3)$	double
2	otherwise	

Table 5: The numbers of backward images.

Here, we remark that the exceptional locus $\widehat{E}(3)$ contains the space $\{(0:0:0:\alpha_1,\alpha_2)\}$, and $\{\mathrm{Discr}(q) = 0\} \cap \{\widehat{E}_0(\boldsymbol{\alpha}) = 0\} = \widehat{E}(3)$.

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